647510 **MSC4MatC402x** **Seat No :________**

M.Sc.(Maths) Semester - 4 (*CBCS*) Examination
May/June-2021 (NEW COURSE)
Integration Theory (CORE)

Time: 1:30 Hours **Marks: 42**

Instructions:

1. Figures to the right indicate marks.
2. There are five questions in the question paper.
3. Answer any three of the following questions.

Unit-1 [14 marks]

Answer **ALL** questions

Q.1 (a) 1 Question of 4 Marks OR 2 Questions of 2 Marks Each. **4 Marks**

(1) Let μ be a measure on a measurable space (X, A). If E and F are measurable subsets of X, $E \subset F$ and if $\mu(F) < \infty$, then show that $\mu(E - F) = \mu(E) - \mu(F)$. 2

(2) Give an example of a measure space (X, A, μ) and a decreasing sequence (E_n) of measurable subsets of X such that $\mu(\bigcap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} \mu(E_n)$. 2

Q.1 (b) Answer any two question out of three. **10 Marks**

(1) Let (X, A) be a measurable space, and let f be a nonnegative measurable function on X. Show that there is an increasing sequence (s_n) of nonnegative measurable simple functions on X such that $s_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$. 5

(2) Let (X, A, μ) be a measure space, and let f be a measurable function on X such that $\int_E f \ d\mu = 0$ for all measurable subset E of X. Show that $f = 0$ a.e. $[\mu]$ on X. 5

(3) Let (X, A, μ) be a measure space. Let (f_n) be a sequence of measurable functions on X converging to a function f pointwise on X. If g is an integrable function on X such that $|f_n| \leq g$ for all n, then show that $\int_X f \ d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$. 5

Unit-2 [14 marks]

Answer **ALL** questions

Q.2 (a) 1 Question of 4 Marks OR 2 Questions of 2 Marks Each. **4 Marks**

(1) Let (X, A, ν) be a signed measure space, and let E and F be positive sets. Show that $E \cup F$ is a positive set. 2

(2) Let (X, A) be a measurable space, and let μ_1, μ_2 and μ be measures on A. If $\mu_1 \perp \mu$ and $\mu_2 \perp \mu$, then show that $(\mu_1 + \mu_2) \perp \mu$. 2

Q.2 (b) Answer any two question out of three. **10 Marks**

(1) Let ν be a signed measure on a measurable space (X, A), and let E be a measurable subset of X such that $0 < \nu(E) < \infty$. Show that E contains a positive set F such that $\nu(F) > 0$. 5

(2) Let ν be a signed measure on a measurable space (X, A). Show that there exist unique measures ν_1 and ν_2 such that $\nu = \nu_1 - \nu_2$ and $\nu_1 \perp \nu_2$. 5

(3) If ν is a signed measure on a measurable space (X, A), then show that X is a disjoint union of a positive set and a negative set. 5

Unit-3 [14 marks]

Answer **ALL** questions

Q.3 (a) 1Question of 4 Marks OR 2 Questions of 2 Marks Each. **4 Marks**

(1) If ν is a finite signed measure on a measurable space (X, A) and if $\alpha \in \mathbb{R}$, then show that $|\alpha\nu| = |\alpha||\nu|$. 2

(2) Let μ be a measure on an algebra A on X, (B_i) be a sequence of elements $A, B \in A$, and let $B \subset \bigcup_i B_i$. Show that $\mu(B) \leq \sum_i \mu(B_i)$. 2

Q.3 (b) **Answer any two question out of three.** **10 Marks**

(1) Let ν be a measure and μ be a σ- finite measure on a measurable space (X, A), and let ν be absolutely continouous with respect to μ. If f is a nonnegative measurable function on X and if E is a measurable subset of X, then show that $\int_E f\, d\nu = \int_E f\, \left[\frac{d\nu}{d\mu}\right] d\mu$. 5

(2) Let ν and μ be σ- finite measures on a measurable space (X, A). Prove that there exists a pair of measures ν_0 and ν_1 such that $\nu_0 \perp \mu$, ν_1 is absolutely continuous with respect to μ and $\nu = \nu_0 + \nu_1$. 5

(3) Let μ^* be an outer measure on X. Show that the collection of all μ^*- measurable subsets of X is a σ- algebra on X. 5

Unit-4 [14 marks]

Answer **ALL** questions

Q.4 (a) **1 Question of 4 Marks OR 2 Questions of 2 Marks Each.** **4 Marks**

(1) Show that every bounded function is essentially abounded. 2

(2) Define Baire measure on the real line. 2

Q.4 (b) **Answer any two question out of three.** **10 Marks**

(1) Let (X, A, μ) be a σ- finite measure space, $1 < p < \infty$, and let q be the conjugate index of p. For $g \in L^q(\mu)$, define $F_g: L^p(\mu) \to \mathbb{R}$ by $F_g(f) = \int_X fg\, d\mu$ for all $f \in L^p(\mu)$. Show that F_g is a continuous linear functional on $L^p(\mu)$ and $\|F_g\| = \|g\|_q$. 5

(2) Let (X, A, μ) and (Y, B, ν) be complete measure spaces. For a measurable rectangle $C \times D$ of $X \times Y$, let $\lambda(C \times D) = \mu(C)\nu(D)$. If $(C_i \times D_i)$ is a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $C \times D$, then show that $\lambda(C \times D) = \sum_i \lambda(C_i \times D_i)$. 5

(3) Let F be a monotone increasing function continuous on the right. If $(a, b] \subset \bigcup_i (a_i, b_i]$, then show that $F(b) - F(a) \le \sum_i (F(b_i) - F(a_i))$. 5

Unit-5 [14 marks]

Answer ALL questions

Q.5 (a) **1 Question of 4 Marks OR 2 Questions of 2 Marks Each.** **4 Marks**

(1) State Riesz- Markov Theorem. 2

(2) Let X be a locally compact Hausdorff space, and let M be a σ-algebra on X containing all Baire sets. When is $E \in M$ called outer regular for μ? 2

Q.5 (b) **Answer any two question out of three.** **10 Marks**

(1) Let μ be a finite measure defined on a σ-algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, then show that it is regular. 5

(2) Let μ^* be a topologically regular outer measure on a locally compact Hausdorff space X. Show that each Borel set is μ^*-measurable. 5

(3) Let μ be a Baire measure on a locally compact space X, and let E be a σ-bounded Baire set in X. If $\epsilon > 0$, then show that there is a σ- compact open set O containing E such that $\mu(O - E) < \epsilon$. 5

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